13. Determine if the following systems has a non-trivial solution:

i. $\begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ -2x_1 - 3x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + 9x_3 = 0 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 9 & 0 \end{array}\right] \underset{-2r_1+r_3}{\overset{2r_1+r_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & -8 & 15 & 0 \end{array}\right] \overset{8r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & -65 & 0 \end{array}\right] \sim$$

$$\overset{-\frac{1}{65}r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ x_2 - 10x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$ trivial solution

→ no free variables ⇒ only trivial solution

ii. $\begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ -3x_2 - 3x_3 = 0 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \underset{\frac{1}{2}r_1}{\overset{2r_1+r_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \underset{-\frac{1}{3}r_2}{\overset{r_2 \leftrightarrow r_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\overset{-r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{cases}$$

free

Sol. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

14. Describe all solutions of $A\bar{x} = \bar{0}$ in parametric vector form, where $A$ is row equivalent to the given matrix. Careful, the matrix below is NOT the augmented matrix, but the coefficient matrix!

i. $\begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{array}\right] \overset{3r_2+r_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = 2x_3 + 7x_4 \\ x_2 = 2x_3 + 4x_4 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 2x_3 + 7x_4 \\ 2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 4x_4 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

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ii. $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\left[\begin{array}{cccccc|c} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{2r_2+r_1}{\sim}$

$$\left[\begin{array}{cccccc|c} 1 & -4 & 0 & 0 & 3 & -7 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{-3r_3+r_1}{\sim} \left[\begin{array}{cccccc|c} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 4x_2 - 5x_6 \\ x_3 = x_6 \\ x_5 = 4x_6 \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2-5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_6 \\ 0 \\ x_6 \\ 0 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

15. Describe and compare the solution sets of $x_1 + 5x_2 - 3x_3 = 0$ and $x_1 + 5x_2 - 3x_3 = -2$.

$x_1 + 5x_2 - 3x_3 = 0$ HOMOGENEOUS ($x_2$ free, $x_3$ free)

$x_1 = -5x_2 + 3x_3$

Sol. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_2+3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$x_1 + 5x_2 - 3x_3 = -2$ NON-HOMOGENEOUS ($x_2$ free, $x_3$ free)

$x_1 = -2 - 5x_2 + 3x_3$

Sol. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-5x_2+3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Solution to $x_1+5x_2-3x_3=-2$ is same as sol. to eq. $x_1+5x_2-3x_3=0$ PLUS vector $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$, $s,t \in \mathbb{R}$

$\to$ translation by $\vec{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

16. Describe the solutions of the following system in parametric vector form, and compare it to that of exercise 12.ii: $\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -4x_1 - 4x_2 - 8x_3 = -16. \\ -3x_2 - 3x_3 = 12 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array}\right] \underset{\substack{\frac{1}{2}r_1 \\ -\frac{1}{3}r_3}}{\overset{2r_1+r_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 \end{array}\right] \overset{r_2 \leftrightarrow r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\overset{-r_2+r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 8 - x_3 \\ x_2 = -4 - x_3 \\ 0 = 0 \end{cases}$$

($x_3$ free)

Sol. $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8-x_3 \\ -4-x_3 \\ x_3 \end{bmatrix} =$

$$= \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}}. \Rightarrow \vec{x} = \vec{p} + t \cdot \vec{v}, \quad t \in \mathbb{R}$$

hom. system in 13. ii had sol. $\vec{x} = t \cdot \vec{v}$, $t \in \mathbb{R}$ $\Rightarrow$ translation by $\vec{p}$.

17. Find the parametric equation of the line through $\vec{a}$ parallel to $\vec{b}$ :

i. $\vec{a}=\begin{bmatrix}-2\\0\end{bmatrix}, \vec{b}=\begin{bmatrix}-5\\3\end{bmatrix}$

$\vec{x} = \vec{a} + t\cdot\vec{b}$ , $t\in\mathbb{R}$ is parameter

$\vec{x} = \begin{bmatrix}-2\\0\end{bmatrix} + t\begin{bmatrix}-5\\3\end{bmatrix}$ or $\begin{cases} x_1 = -2-5t \\ x_2 = 3t \end{cases}$

ii. $\vec{a}=\begin{bmatrix}3\\-2\end{bmatrix}, \vec{b}=\begin{bmatrix}-7\\6\end{bmatrix}$

$\vec{x} = \vec{a} + t\vec{b}$ , $t\in\mathbb{R}$ parameter

$\vec{x} = \begin{bmatrix}3\\-2\end{bmatrix} + t\begin{bmatrix}-7\\6\end{bmatrix}$ or $\begin{cases} x_1 = 3-7t \\ x_2 = -2+6t \end{cases}$

18. Does the equation $A\vec{x}=\vec{0}$ have a non-trivial solution, and does the equation $A\vec{x}=\vec{b}$ have at least one solution for every possible $\vec{b}$ ?

i. $A$ is a $3\times3$ matrix with 3 pivot positions

$A = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$

$\hookrightarrow$ no free variables

ⓐ $A\vec{x}=\vec{0}$ has only trivial solution $\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

ⓑ $A\vec{x}=\vec{b}$ has exactly one solution $\forall$ vector $\vec{b}$ (by Thr. 4, $\forall\, b\in\mathbb{R}^3$ $A\vec{x}=\vec{b}$ has a sol.)

ii. $A$ is a $4\times4$ matrix with 3 pivot positions

$A = \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\uparrow$ free variable

ⓐ $A\vec{x}=\vec{0}$ has a free variable

$\vec{x} = \begin{bmatrix}*\\*\\*\\x_4\end{bmatrix} \Rightarrow \infty$-many solutions

ⓑ $A\vec{x}=\vec{b}$ does not have a sol. $\forall\, \vec{b}\in\mathbb{R}^4$ (by Thr. 4)

$[0\,0\,0\,0 \,|\, *]$ gives inconsistent system

$\uparrow$ non-zero

iii. $A$ is a $2\times5$ matrix with 2 pivot positions

$A = \begin{bmatrix} 1 & * & 0 & * & * \\ 0 & * & 1 & * & * \end{bmatrix}$

free variables

ⓐ $A\vec{x}=\vec{0}$ has 3 free variables

$\vec{x} = \begin{bmatrix}*\\x_2\\*\\x_3\\x_4\end{bmatrix} \Rightarrow \infty$ many Solutions

ⓑ $A\vec{x}=\vec{b}$ has at least 1 solution $\forall\, b\in\mathbb{R}^2$ (by Thr. 4 it has a pivot position in every row).

19. Mark each statement as ***True*** or ***False*** and justify your answer.

i. An example of a linear combination of vectors $\vec{v}_1$ and $\vec{v}_2$ is the vector $\frac{1}{2}\vec{v}_1$. True

ii. The weights $c_1$, $c_2$, ... $c_p$ in a linear combination $c_1\vec{v}_1 + ... + c_p\vec{v}_p$ cannot all be zero. False

$\vec{0} \in \text{Span}\{\vec{v}_1, ..., \vec{v}_p\}$

iii. The equation $\vec{x} = \vec{p} + t\vec{v}$ describes a line through $\vec{v}$ parallel to $\vec{p}$. False

line parallel to $\vec{v}$ through $\vec{p}$

iv. The equation $A\vec{x} = \vec{b}$ is referred to as a vector equation. False.

Matrix equation

v. If $A$ is a $m \times n$ matrix and if the equation $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$ in $R^m$, then $A$ cannot have a pivot position in every row. True (see Thm. 4 / p.37)

vi. The solution set of a linear system whose augmented matrix is $[a_1 \quad a_2 \quad a_3 \quad b]$, is the same as the solution set of $A\vec{x} = \vec{b}$ if $A = [a_1 \quad a_2 \quad a_3]$. True

vii. If the equation $A\vec{x} = \vec{b}$ is consistent, then $\vec{b}$ is in the set spanned by the columns of $A$. True